The threefold classification of unstable disturbances in flexible surfaces bounding inviscid flows

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This paper discusses the general idea that in systems where a flexible solid is coupled with a flowing fluid three different types of instability are possible. These were originally designated by Brooke Benjamin (1960) as 'class A', 'class B' and 'Kelvin–Helmholtz' instability, and their collective significance has been clarified recently by Landahl (1962). Class A and class B disturbances are essentially oscillations involving conservative energy-exchanges between the fluid and solid, but their stability is determined by the net effect of irreversible processes, which include dissipation and energy-transfer to the solid by non-conservative hydrodynamic forces. Dissipation in the solid tends to stabilize class B disturbances but to destabilize class A ones. Class C instability (i.e. the 'Kelvin–Helmholtz' type) occurs when conservative hydrodynamic forces cause a unidirectional transfer of energy to the solid.

In § 2 this idea is examined fundamentally by way of the Lagrangian method of generalized co-ordinates, and in § 3 the example of inviscid-fluid flow past a flexible plane boundary is considered. The treatment of this example amplifies the work of Landahl, in particular by including the effect of non-conservative forces of the kind investigated by Miles in his series of papers on water-wave generation by wind.

1. Introduction

In the analysis by Brooke Benjamin (1960) of the stability of laminar boundary layers over flexible surfaces, it was shown that three categories of instability may be distinguished, which were named class A, class B, and Kelvin–Helmholtz instability, respectively. In a recent paper on the same problem Landahl (1962) has considerably elucidated this scheme of classification by providing a simple physical interpretation of the distinctive factors among the three types of unstable disturbance, and he has shown that the underlying principles are not restricted to the particular problem of viscous boundary-layer instability in which they first came to light. Specifically, he showed them still to apply when the fluid is inviscid and the flow is uniform without any boundary layer. In the present paper these principles will be explored further on the simple lines which have already led to Landahl's important discoveries.

A remarkable property of class A disturbances is that dissipation in the solid has a destabilizing effect on them. In view of the claim by Krämer (1960) that his method of 'distributed damping' provided boundary-layer stabilization, this property was discovered with some surprise in the treatment of the corresponding theoretical problem, where the class A disturbances were identified as Tollmien–Schlichting waves modified by the response of the flexible boundary to the attendant pressure fluctuations. But Landahl's investigation has shown that the latter interpretation is incidental and the destabilizing influence of dissipation is common to the simpler example of class A disturbances with which he illustrated their fundamental nature. According to the general definition to be emphasized in this paper, both class A and class B disturbances are oscillations which, in the absence of irreversible processes such as dissipation and the conversion of energy by Reynolds stresses, would be sustained at constant amplitude. The distinctive attribute of class A oscillations is that their development entails a reduction, proportional to amplitude squared, in the total-energy level of the complete system; and since a loss of energy by the system can therefore be compensated by an increase of amplitude, this readily explains the way in which dissipation affects them.

[The stability problem for laminar boundary layers over flexible surfaces has also been studied theoretically by Betchov (1960), Boggs & Tokita (1960) and Hains & Price (1962), who appear to have considered only the class A wave modes, and by Nonweiler (1961) who dealt more comprehensively with particular examples of a non-dissipative surface. In the related context of panel flutter, cases where structural damping has an unfavourable effect on stability have been demonstrated by Hedgepeth, Budiansky & Leonard (1954), Nelson & Cunningham (1955), Johns & Parks (1960) and others.]

The fundamental property of class B oscillations is that their initiation entails a raise in energy level,† the opposite of the previous case. Accordingly the attenuating effect of dissipation on them is easily understood.† However, these oscillations tend to be amplified by irreversible energy-transfer (i.e. brought about by non-conservative forces) from the fluid to the solid, and such a process leaves the total-energy level unchanged. In general, therefore, the total energy relative to the quiescent state is not a suitable measure of the extent to which the system is disturbed, except perhaps in a period immediately following the initiation of a disturbance before a significant amount of energy is converted by the irreversible processes. To cover this aspect we shall introduce the concept of the 'activation energy', which is the instantaneous level of the conservative energy-exchanges associated with class A or class B oscillations. As an oscillation amplifies or attenuates, the activation energy departs from the level of the initial excitation by an amount which represents the balance of energy converted irreversibly by the disturbance. This concept is apparently new, but it seems indispensible to a simple collective interpretation of class A and class B properties.

In their treatments of the problem of boundary-layer stability, Brooke Benjamin and Landahl identified the class B disturbances with 'free surface waves' which could still propagate along the boundary if the fluid were absent,

† This compares with the familiar case of stable oscillations in a finite conservative system whose quiescent state has zero kinetic energy and minimum potential energy; but note that the present case is special in that the quiescent state of a system including a flow may have unbounded kinetic energy.

but which are modified by the reaction of the disturbed flow. These waves pose essentially the same theoretical problem as the one formulated by Miles (1957, 1959, 1962*a*, *b*) to explain the generation of water waves by wind. His theory shows that when an inviscid shear flow is disturbed by a sinusoidal wave travelling along the boundary in the same direction, the pressure perturbation at the boundary can have a component in phase with the wave slope; this arises when there exists above the boundary a 'critical height' at which the flow velocity equals the wave velocity, and the curvature of the flow-velocity distribution with height is negative there. Work is done on the moving surface by this pressure component, and so a wave will be sustained at constant amplitude on a dissipative boundary (specifically a water surface subject to viscous stresses in Miles's model) if the rate of energy-transfer across it balances the rate of dissipation. The energy supply derives, of course, from the kinetic energy of the mean shear flow, the process of abstraction being associated with a constant Reynolds stress at all heights up to the critical. A very informative physical interpretation of Miles's theory has been given recently by Lighthill (1962), who explained the energy-transfer mechanism in terms of processes operating in the region of the critical height.

Although the case of 'neutral stability' described above has been the only one treated precisely by Miles, and also by Lighthill, a crucial implication of his theory is that a wave will grow if the dissipation is insufficient to balance the energy-transfer to a corresponding neutral wave. While being clearly correct, of course, for Miles's particular model of a water–air interface, this interpretation of neutral stability is by no means obviously forthcoming in a more general view of this type of problem. Indeed, just the opposite interpretation holds for class A waves, and even for class B ones the exact significance of the energy-transfer according to Miles's analysis is not immediately evident in other than the neutral case.

To be specific about the latter point, we recall that the rate of energy-transfer to a neutral wave is $\dot{W}_n = \overline{cp_s \partial\eta/\partial x}$ per unit surface area (see § 3 for notation; the suffix n here implies evaluation of $\dot{W}$ for c real), and if D is the dissipation per unit area then $\dot{W}_n = D$ is the condition of neutral stability which justifies exactly the basis for the calculation of $\dot{W}_n$. But, with the same $\dot{W}_n$, Miles's theory proceeds to use the approximation that $\dot{W}_n - D\ (>0)$ is the rate of accumulation of energy by the water wave, although strictly the pressure component in phase with the wave elevation (comprising the conservative part of the hydrodynamic forces) also does some work on a *growing* wave, so that $\dot{W}_n$ is not the complete energy-transfer.† It has been pointed out by Miles (1962*b*, p. 81) that the error in this approximation is of the same order of magnitude as the ratio of the air and water densities, and so is negligible. But what is the *precise* physical significance of $\dot{W}_n - D$ in this case when $\dot{W}_n$ still has the distinctive meaning accorded by Miles's analysis? And how does $\dot{W}_n - D$ relate to the wave growth when the densities of the two parts of the system are comparable, so that the approximation in question breaks down? This basic aspect of Miles's theory will be clarified in the present paper.

† This approximation was also used in Jeffreys's 'sheltering theory' of water-wave generation as reported by Lamb (1932, § 348).

The third type of instability, to be called here class C, was previously named after Kelvin and Helmholtz whose analyses of the stability of contiguous uniform streams with unequal velocities gave the first example of it (Lamb 1932, §§ 232, 268). Unlike the first two, this type of instability is virtually independent of irreversible effects in the system if they are small. It is thus a consequence of the conservative forces acting on a small disturbance, being analogous to the instability of a conservative system in static equilibrium when the potential energy is a maximum. In the case of flow past a plane boundary, class C instability always arises ultimately if the flexibility of the boundary is made large enough, specifically so that the hydrodynamic suction in phase with a wavy deformation outweighs the restoring forces for all values of the wave speed. Analogous examples in other solid-fluid systems include the buckling of flexible pipes conveying fluid (Brooke Benjamin 1961) and the yawing instability of a towed flexible body (Hawthorne 1961).

The contribution of this paper is in two parts. First, in § 2, the three types of instability are illustrated in a general way by means of the Lagrangian method applied to an unspecified system, wherein a flexible solid is initially in equilibrium under the influence of an infinite steady flow of inviscid fluid. Here the principal aim is to emphasize that instabilities of many systems other than those with a plane solid–fluid interface may be classified in the same way. For instance, the generalized theoretical model includes the case of towed finite bodies like the 'Dracone' flexible oil-barge (Hawthorne 1961) and also systems of elastic pipes, or articulated rigid ones, through which fluid is pumped (Brooke Benjamin 1961). A hypothetical 'normal mode' of disturbance from equilibrium is considered, thus requiring only a single generalized co-ordinate to represent the energy of the solid and the energy-transfer from the fluid, and so the analysis is extremely simple. This approach is intended merely to provide an overall interpretation of possible behaviour in a variety of systems, and of course it evades the far more demanding task of formulating and solving the complete dynamical equations for any particular system.

In § 3 the problem of semi-infinite inviscid flow past a flexible plane boundary is reconsidered on the lines of Landahl's discussion (1962, § 7). The main points of his analysis are covered again with only minor elaborations, the importance of these points being considered sufficient to justify this recapitulation, but one significant modification is introduced. Although the direct effects of viscosity on a wavy disturbance are again ignored, the flow is assumed to feature a thin boundary layer which gives rise to an irreversible energy-transfer mechanism of the kind discovered by Miles. It will be definitely established, confirming Landahl's prediction, that this mechanism has a destabilizing influence only on class B waves and, rather surprisingly, it tends to stabilize the class A waves which can also exist in this type of system.

2. Illustration in terms of generalized dynamics

We consider a generic system in which a slightly dissipative flexible solid is coupled with an infinite flow of frictionless fluid. From an initial state of equilibrium the system is disturbed infinitesimally in a particular normal mode, the magnitude of the deformation being measured as a function of time by a generalized co-ordinate $q(t)$. For the disturbance in the solid, the kinetic energy, potential energy and Rayleigh dissipation function are given respectively by

$$T = \tfrac{1}{2}\mu\dot{q}^2, \quad V = \tfrac{1}{2}\lambda q^2, \quad R = \tfrac{1}{2}\kappa\dot{q}^2, \tag{2.1}$$

where the inertia, stiffness and friction coefficients μ, λ and κ are positive constants. The Lagrangian dynamical equation for the solid is therefore

$$\mu\ddot{q} + \kappa\dot{q} + \lambda q = Q, \tag{2.2}$$

where Q is the 'generalized-force component' which represents the action of the fluid upon the solid for the normal mode in question (cf. Brooke Benjamin 1961, p. 464).

For the purpose of illustration it is supposed that Q takes the form

$$Q = M\ddot{q} + K\dot{q} + \Lambda q, \tag{2.3}$$

in which M, K and Λ are real constants. This assumption is simply made *ad hoc* to provide the properties demonstrated below, but there appear to be many examples which bear it out (see, for example Brooke Benjamin 1961, §§ 2·6, 3·2). For travelling-wave disturbances, however, Q takes a slightly different form and the modification will be explained later.

It is worth noting that a non-conservative part of the hydrodynamic forces, as represented by the term with coefficient K in (2.3), is a common feature of systems in which a *finite* flexible solid is coupled with an infinite flow and is free to make lateral displacements at its downstream end (cf. the analysis by Lighthill (1960) of a slender fish's swimming motions; also Brooke Benjamin (1961, § 2·3)). It has already been noted in § 1 that, by a quite different process, wavy disturbances of a shear flow also give rise to non-conservative forces.

Now, if the disturbance is initiated by external forces applied impulsively *to the solid*, say just after $t = 0$, the total work done on the solid by the hydrodynamic forces is given by

$$W = \int_0^t Q\dot{q}\,dt = \tfrac{1}{2}M\dot{q}^2 + \tfrac{1}{2}\Lambda q^2 + K\int_0^t \dot{q}^2\,dt, \tag{2.4}$$

since $q(0) = 0$ and $\dot{q}(0) = 0$. This must also be the energy lost by the fluid (from the unbounded store of kinetic energy possessed by the primary flow), so that

$$\begin{aligned}\mathscr{E} &= T + V - W \\ &= \tfrac{1}{2}(\mu - M)\,\dot{q}^2 + \tfrac{1}{2}(\lambda - \Lambda)\,q^2 - K\int_0^t \dot{q}^2\,dt\end{aligned} \tag{2.5}$$

is the total energy of the whole system relative to the original quiescent state. Energy may be dissipated only in the solid since the fluid is frictionless, and so we have $d\mathscr{E}/dt = -2R$ ($\leqslant 0$) or, upon integration,

$$\mathscr{E} = \mathscr{E}_0 - \kappa\int_0^t \dot{q}^2\,dt, \tag{2.6}$$

where $\mathscr{E}_0$ is the energy level immediately after the initial excitation.

As compared with the total energy, which is not directly changed by the irreversible energy-transfer proportional to K, a more useful measure of the degree of excitation is what we may suitably term the 'activation energy' E, which is the sum of $\mathscr{E}$ and the energy transferred to the solid by the non-conservative hydrodynamic forces. Thus we consider

$$\begin{aligned} E &= \mathscr{E} + K\int_0^t \dot{q}^2\,dt \\ &= \tfrac{1}{2}(\mu - M)\,\dot{q}^2 + \tfrac{1}{2}(\lambda - \Lambda)\,q^2. \end{aligned} \tag{2.7}$$

According to (2.7) E is also interpretable as the energy, relative to that of the quiescent system, which is involved in *conservative* exchanges between kinetic and potential forms (and between the two parts of the system) during an oscillatory motion. Finally, combining (2.6) and (2.7), we get the equation

$$E - \mathscr{E}_0 = (K - \kappa)\int_0^t \dot{q}^2\,dt, \tag{2.8}$$

the right-hand side of which is the difference between the non-conservative energy-transfer to the solid and the dissipation within it, and so represents the balance of energy converted *irreversibly* by the disturbance. Note, however, that this is not the actual gain of energy by the solid, nor the average gain during an oscillation of gradually varying amplitude, because obviously the conservative hydrodynamic forces may also contribute to the energy-transfer. [Equation (2.8) is, of course, a formal consequence of (2.2) and (2.3), being given immediately by integration after multiplication by $\dot{q}$. But the present indirect derivation serves to prepare a comprehensive physical explanation for the properties of the system.]

When the dynamical equation given by combining (2.2) and (2.3) is solved, the following three cases may be distinguished. We assume that κ and K are both small in comparison with $|(\mu - M)(\lambda - \Lambda)|$.

(i) $\mu > M$ *and* $\lambda > \Lambda$

In the absence of dissipation and of non-conservative hydrodynamic forces ($\kappa = 0$ and $K = 0$), the solution describes a simple-harmonic motion at the frequency

$$\omega = \left(\frac{\lambda - \Lambda}{\mu - M}\right)^{\frac{1}{2}}, \tag{2.9}$$

and one finds that $\mathscr{E} = E = \frac{1}{2}(\lambda - \Lambda)\,\hat{q}^2$, where $\hat{q}$ is the amplitude. Thus the total-energy level is positive.

For finite yet small κ and K the frequency is little changed, but the oscillation is damped or amplified accordingly as $\kappa \gtrless K$. The condition of instability is therefore that $K > \kappa$, which of course means that the rate of irreversible energy-transfer from the fluid to the solid exceeds the mean rate of dissipation. If its direction is changed ($K < 0$), the irreversible energy-transfer has a damping effect additional to that of the dissipation.

The instability for $K > \kappa$ is readily explained in physical terms by reference to (2.8). The activation energy, which is given very closely by $E = \frac{1}{2}(\lambda - \Lambda)\,\hat{q}^2$, must be positive to begin with (i.e. a positive amount of energy $\mathscr{E}_0$ must be added to the system in generating the disturbance), and if $K > \kappa$ it steadily increases even though the total energy $\mathscr{E}$ steadily decreases according to (2.6). The energy of the initial excitation, $\mathscr{E}_0$, is eventually lost by the system and $\mathscr{E}$ becomes negative; but the disturbance grows because this loss is more than compensated by the transfer to the disturbance of energy from the infinite store in the fluid.

This case exemplifies class B instability in all essential respects, and the instability criterion clearly corresponds to the criterion for wave growth in Miles's theory.

(ii) *$\mu < M$ and $\lambda > \Lambda$, or $\mu > M$ and $\lambda < \Lambda$*

The solution is

$$q = a \sinh \sigma t + b \cosh \sigma t, \tag{2.10}$$

with $\sigma^2 \doteqdot (\Lambda - \lambda)/(\mu - M) > 0$, and now κ and K have no qualitative effect on the solution if they are small. Thus, the system is vigorously unstable irrespective of the action of the non-conservative forces.

For κ and K small the activation energy is very nearly

$$E = \tfrac{1}{2}(\Lambda - \lambda)(a^2 - b^2). \tag{2.11}$$

This can be either positive or negative, but the more important fact is that it does not vary as the instability develops. The physical interpretation of the instability is that there is a unidirectional transfer of energy from the fluid to the solid, being effected by conservative forces and so leaving the activation energy unchanged. Though as before the total-energy level will be gradually reduced by dissipation, this factor is now unimportant since the redistribution of energy between the two parts of the system occurs much more rapidly.

This case typifies class C (or 'Kelvin–Helmholtz') instability.

(iii) *$\mu < M$ and $\lambda < \Lambda$*

For $\kappa = 0$ and $K = 0$ the solution again describes a simple-harmonic motion at the frequency ω given by (2.9), but now the energy level of the disturbance is $\mathscr{E} = -\tfrac{1}{2}(\Lambda - \lambda)\hat{q}^2$ and so is negative. This means simply that the absolute energy level of the whole system must be reduced in the process of creating a free oscillation: that is, the system must be allowed to do work against the external forces which provide the initial excitation.

For small finite κ and K the oscillation is amplified if $\kappa > K$ and damped if $\kappa < K$. Thus dissipation and irreversible energy-transfer from the fluid to the solid have opposite effects in this case as compared with (i). In particular, the effect of dissipation is always destabilizing.

An interpretation of the physical mechanism of instability is again indicated by (2.8). The activation energy $E \doteqdot -\tfrac{1}{2}(\Lambda - \lambda)\hat{q}^2$ is negative when a disturbance is first created (i.e. $\mathscr{E}_0 < 0$), and the amplitude of the oscillation is made progressively larger by increases in the negative magnitude of E, which occur when $\kappa > K$. The significance of E in the present case is perhaps made clearest as follows. Suppose that the irreversible processes were suddenly stopped so that the oscillation continued at constant amplitude $\hat{q}$. Then clearly E is what the *absolute* energy level of the system would be if the same oscillation had been excited by external forces, and we know from two paragraphs above that E as thus defined is essentially negative, increasing in magnitude with q. Hence we readily appreciate that dissipation is destabilizing since it lowers the absolute energy level. Again, irreversible energy-transfer to the solid ($K > 0$) tends to stabilize since according to the definition (2.7) it raises E above the absolute energy level, so bringing the 'excitation level' closer to the level of the quiescent system.

This case exemplifies class A instability in all essential respects.

Though the preceding theoretical model provides the simplest demonstration of essentials, there are evidently many cases to which it does not apply precisely yet which admit the same general interpretation regarding the classification of instabilities. We now note an extension of our Lagrangian formulation to a category of travelling-wave disturbances which includes the case to be considered in § 3.

Suppose that the system is both uniform and unbounded in the flow direction x, and that it is disturbed by a sinusoidal wave travelling in this direction. The motion within a fixed interval of x may be considered to comprise two modes $q = q_1(t)\cos\alpha x$ and $q = q_2(t)\sin\alpha x$, in which q_1 and q_2 are oscillations in quadrature. In the absence of the fluid q_1 and q_2 would independently satisfy the same Lagrangian equation, but through the action of the flow there may be coupling between the two modes. To represent this effect q is taken to be complex, on the understanding that the real part of $q\,e^{i\alpha x}$ describes the physical disturbance. As before the Lagrangian equation for the solid under the influence of the flow is

$$(\mu - M)\ddot{q} + (\kappa - K)\dot{q} + (\lambda - \Lambda)q = 0, \tag{2.12}$$

but whereas μ, κ, λ, M and Λ are again real constants, K may now be complex. (We can assume that in general M will be real and *negative* in view of the fact that $-M$ is the virtual-mass coefficient for small displacements in the modes considered. And it can be assumed that the hydrodynamic pressure or suction on the solid will be in phase with *static* displacements in these modes, which means that Λ will be real.)

Corresponding to (2.4), the energy-transfer W averaged over x is given by the real part of the integral of $\frac{1}{2}Q^*\dot{q}$, where Q^* is the complex conjugate of Q. The term $iK_i\dot{q}$ in Q therefore makes no contribution to W, and so the expressions for the energies $\mathscr{E}$ and E are the same as before except that K is replaced by K_r. Thus dE/dt takes the sign of $K_r - \kappa$.

Representing the two solutions of (2.12) in the form $q\,e^{-i\nu t}$, we get

$$\nu = \frac{K_i}{2(\mu - M)}\left[-1 - \frac{i(\kappa - K_r)}{K_i} \pm \sqrt{\left\{1 + R + \frac{2i(\kappa - K_r)}{K_i}\right\}}\right], \tag{2.13}$$

where
$$R = \{4(\mu - M)(\lambda - \Lambda) + (\kappa - K_r)^2\}/K_i^2.$$

Instability is indicated by ν having a positive imaginary part.

The following three cases may now be distinguished: (1) When $R > 0$, both solutions are subject to the same instability condition $K_r > \kappa$ (i.e. $dE/dt > 0$), and so both are class B. (2) When $-1 < R < 0$, one solution is again class B but the other is class A, being unstable if $\kappa > K_r$. (3) When $R < -1$, the radical in (2.13) has a large imaginary part so that instability of the class C (or 'Kelvin–Helmholtz') type occurs.

This situation will be examined more fully in the example which follows.

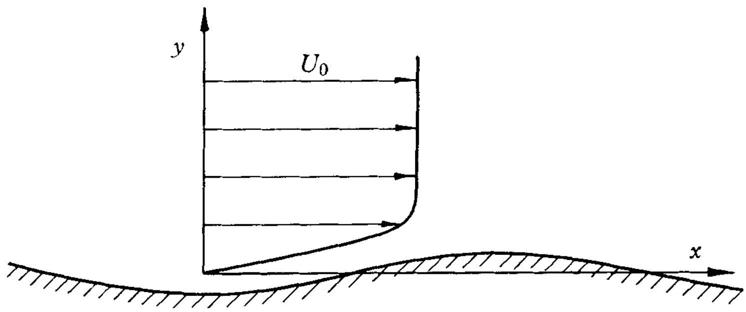

FIGURE 1. Definition sketch showing the undisturbed velocity profile.

3. Stability of a plane interface between a flexible solid and a parallel flow of inviscid fluid

Considering a travelling-wave disturbance, we take the equation of the deformed interface (see figure 1) to be

$$y = \eta(x,t) = \mathrm{Re}\{\hat{\eta}\, e^{i\alpha(x-ct)}\}, \tag{3.1}$$

and regard $\alpha\hat{\eta}$ as infinitesimal. Our object is to find the complex wave-velocity $c = c_r + ic_i$ corresponding to a given real wave-number α, and in particular to examine the possibilities of the condition $c_i > 0$ indicative of instability. A relationship for c will be obtained by equating the pressure p_s exerted by the fluid on the interface and the normal stress generated in the solid by the deformation, the dynamic resistance of the solid boundary being represented by three gross parameters. No shear stress is exerted by the inviscid fluid, and the condition of vanishing shear stress at the solid boundary does not enter the present analysis explicitly, although it is evidently a factor upon which the resistance parameters might depend.

When the primary flow is taken to have a *uniform* velocity U_0, the surface pressure is found by irrotational-flow theory to be

$$p_s = -\rho(U_0-c)^2\alpha\eta, \tag{3.2}$$

where ρ is the fluid density. (The weight of the fluid is ignored in the present analysis, which is very well justified if the fluid is air; but otherwise, if the interface is horizontal, there is merely an addition $-\rho g\eta$ to (3.2).) To make a rather more realistic model, however, we suppose there to be a thin boundary layer over which the primary velocity $U(y)$ increases steadily from zero at the undisturbed interface $y = 0$ and approaches the value U_0 asymptotically, but we assume that the displacement thickness δ^* is much smaller than the wavelength

$2\pi/\alpha$. Then (3·2) is a good approximation to the component of pressure in phase with the wave elevation (Brooke Benjamin 1959, § 7), but if $0 < c < U_0$ there is also a component in phase with the wave slope. This second pressure component arising from non-uniformity of the flow has central importance in Miles's theory (1957, 1959, 1962 *a*, *b*) of water-wave generation by wind, which considers that for a neutral wave ($c_i = 0$) the mean energy-transfer to the water surface is proportional to it. We accordingly write

$$p_s = \rho\{-(U_0-c)^2+iSc\}\alpha\eta; \tag{3.3}$$

and since the second component is comparatively small (see below) we shall treat S as a parameter in the relationships derived to determine c.

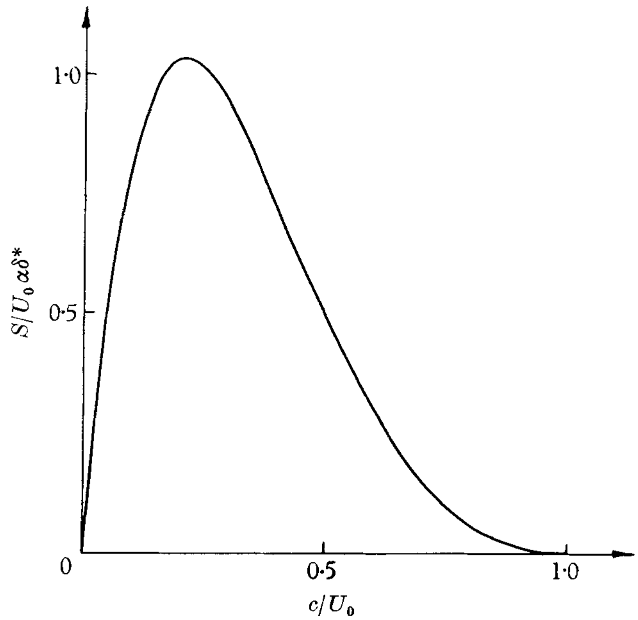

FIGURE 2. Values of the dimensionless ratio $S/U_0\alpha\delta^*$ for a laminar boundary layer with zero pressure gradient.

For a thin boundary layer and c real it can be shown (see Brooke Benjamin 1959, equation (7.32)) that an approximation to Sc of the same standing as the present approximation to the in-phase component is

$$Sc = -\pi\alpha(U_0-c)^4\, U_c''/U_c'^3, \tag{3.4}$$

where U_c' and U_c'' are derivatives of $U(y)$ evaluated at the 'critical height' where $U = c$. Hence the ratio $Sc/(U_0-c)^2$ of the magnitudes of the terms in (3.3) is deduced to be $O(\alpha\delta^*)$ and thus very small according to the assumption made above. Figure 2 gives values of $S/U_0\alpha\delta^*$ calculated from (3.4) for the laminar boundary layer along a plane surface (Schlichting 1955, Ch. VII, § *f*) and plotted as a function of c/U_0. When c is complex, as will be considered presently, this approximation to the pressure component arising from the boundary layer is still good if c_r is substituted for c, provided that $c_i \ll c_r$.

The normal reaction of the solid boundary to the deformation (3.1) may be represented as the sum of a resilient, an inertial and a frictional component; thus, for the direct stress acting inwards (the direction of negative η), we put

$$-\left(F\alpha^2\eta + m\frac{\partial^2\eta}{\partial t^2} + km\alpha\frac{\partial\eta}{\partial t}\right) = -(F\alpha - m\alpha c^2 - ikm\alpha c)\,\alpha\eta. \tag{3.5}$$

Here F denotes the tension per unit span of an equivalent membrane, m the equivalent mass per unit surface area, and $km\alpha$ the frictional resistance. This representation is valid for a variety of systems, perhaps the simplest being a membrane under tension and a thin plate ($F = \alpha^2 \times$ *flexural rigidity*). [It also applies to the surface of deep water, provided the effects of surface currents caused by wind are neglected (as they were by Miles). For this case one takes $F = \rho_w g/\alpha^2$, $m = \rho_w/\alpha$ and $k = 4\nu_w\alpha$, where ρ_w is the density and ν_w the kinematic viscosity of water. For the corresponding case where the density ρ of the fluid above the interface is comparable with the density ρ_w of a fluid below, inclusion of the weight of the upper fluid leads to a form of (3.5) with

$$F\alpha^2 = (\rho_w - \rho)\,g.]$$

Note that the quantity within the parentheses on the right-hand side of (3.5) is equivalent to the dynamic-stiffness coefficient β defined by Brooke Benjamin (1960), and to $-ic/Y$ in the notation used by Landahl (1962).

The dynamical equation for the disturbance is obtained by equating (3.3) and (3.5). Putting $c_0^2 = F/m$, we get

$$(U_0 - c)^2 - iSc = \zeta(c_0^2 - c^2 - ikc), \tag{3.6}$$

where $\zeta = m\alpha/\rho$ is a dimensionless ratio. Note that in the absence of the fluid and of friction in the solid the solution of (3.6) is $c = \pm c_0$. Thus c_0 is the speed of free surface waves in the solid.

Leaving the dependence of S on c implicit, we may regard (3.6) as a quadratic in c, the solution of which is

$$c = \frac{1}{\zeta+1}\left[U_0 - \tfrac{1}{2}i(\zeta k - S) \pm \sqrt{\{\zeta(\zeta+1)\,c_0^2 - \zeta U_0^2 - \tfrac{1}{4}(\zeta k - S)^2 - i(\zeta k - S)\,U_0\}}\right]. \tag{3.7}$$

Since S is very much smaller than U_0, an accurate explicit approximation to c may be obtained from (3.7) by evaluating S at the figure for c given when $S = 0$ in this expression. We shall assume for simplicity that ζk also is small in comparison with U_0 (i.e. frictional resistance $\ll \rho U_0$).

Following Landahl (1962) we may distinguish three cases in which (3.7) provides different interpretations:

$$\left.\begin{array}{ll}\text{(i)} & U_0^2 < \zeta c_0^2,\\ \text{(ii)} & \zeta c_0^2 < U_0^2 < (\zeta+1)\,c_0^2,\\ \text{(iii)} & U_0^2 > (\zeta+1)\,c_0^2.\end{array}\right\} \tag{3.8}$$

It should first be noted that a vigorous instability of the class C, Kelvin–Helmholtz type occurs in case (iii), because then the radical in (3.7) is mainly imaginary (becoming purely imaginary if $k = 0$ and $S = 0$) and so one solution has a large positive imaginary part c_i.

To illustrate cases (i) and (ii) the two values of c given by (3.7), say a and b, may be approximated to the first order in $(\zeta k - S)$, which is taken to be small in comparison with u defined below. Thus we consider

$$a, b = \frac{1}{\zeta+1}\left[U_0 \mp u - \tfrac{1}{2}i(\zeta k - S_{a,b})\left(1 \mp \frac{U_0}{u}\right)\right], \tag{3.9}$$

where $$u = \surd\{\zeta(\zeta+1)\,c_0^2 - \zeta U_0^2\}.$$

The alternative signs are minus for a and plus for b, and it is understood that S is evaluated at $c = a_r$ and $c = b_r$ respectively.

In case (i) we have that $u > U_0$, and so $a_r < 0$. For a wave thus travelling in the direction opposite to the flow, there is no 'critical height' and consequently $S = 0$; hence the disturbance is damped if $k > 0$. The wave with $c = b$ is damped only if $\zeta k > S_b$, and it appears that the interface is unstable if $S_b > \zeta k$. This instability clearly is of the class B type.

[Note that if ζ is very large, yet U_0/c_0 is $O(1)$, (3.9) gives

$$b_r \doteqdot c_0 - (2\,\zeta c_0)^{-1}(U_0 - c_0)^2,$$

which accords with one of the conditions assumed in Miles's theory. He supposed that a water wave acted upon by moderate wind has a speed very close to the value $c_0 = (g/\alpha)^{\frac{1}{2}}$ for a wave on a free surface, and this assumption is amply justified—at least for wind speeds not much in excess of c_0—by the fact that $\zeta = \rho_w/\rho \doteqdot 800$ for his model. The condition for wave growth considered by Miles is equivalent to the present instability condition $S_b > \zeta k$, and it is worth noting that the wind speed U_0 must exceed c_0 to give a non-zero S and so make wave growth possible. We note also that the approximation to the rate of growth used by Miles and by Jeffreys, as was mentioned in § 1, is equivalent to the approxmation $b_i = \frac{1}{2}(\zeta^{-1}S_b - k)$ obtained from (3.9) when $\zeta \gg 1$ and so $u \gg U_0$.]

In case (ii) we have that $0 < u < U_0$, and so both a_r and b_r are positive. The instability condition for the wave with $c = b$ is again that $S_b > \zeta k$, but now a new feature arises in that the slower-travelling wave has class A properties. For we see from (3.9) that the wave is unstable if $\zeta k > S_a$, which means that friction and Miles's out-of-phase pressure component have exchanged their previous roles, the former now being destabilizing and the latter stabilizing. Since $S \to 0$ for $c_r \to 0$ (see figure 2), the condition for incipience of class A instability as the flow velocity is gradually raised in a dissipative system is that

$$U_0^2 = \zeta c_0^2, \quad \text{i.e.} \quad \rho U_0^2 = F\alpha, \tag{3.10}$$

which gives $a_r = 0$. This condition means that the hydrodynamic pressure is just sufficient to maintain a stationary wave against the resilient restoring forces, and so it corresponds to the condition of 'static divergence' commonly propounded in studies of panel flutter; but an important point revealed here is that (3.10) represents a stability limit *only* if the system is dissipative (cf. Landahl 1962, p. 629). The value of U_0^2 at which class C instability arises is greater than the value (3.10) by a fraction $\zeta^{-1} = \rho/m\alpha$, and so in cases where ζ is large the range of flow speeds over which class A instability can occur may be insignificant compared with the 'Kelvin–Helmholtz' limit. For instance, it is generally unlikely to be significant when the fluid is air, although one must still allow for the possibility of waves strongly coupled with an air flow in consequence of having speeds and wavelengths close to those of Tollmien–Schlichting waves, which are of class A type (see Miles 1962*a*). When the fluid is water, however, there is likely to be a considerable margin between the limiting conditions for class A and class C instability, so that the fomer is distinctly the crucial factor in determining a practical criterion of stability.†

Just as in the generalized case treated in § 2, a neat physical interpretation of these results is forthcoming when the energy of the system is considered. A similar investigation was made by Landahl (1962), but now a slightly different approach is necessary to account for the part which the pressure component proportional to S plays in the energy balance. Proceeding as in § 2 we have to express the 'activation energy' which, since k and S are small, is practically the same as the total-energy level for a wave of amplitude $\hat{\eta}$ in a corresponding conservative system, i.e. with $k = S = 0$ (see equation (2.7)). For the solid the sum of the mean kinetic energy and mean potential energy per unit area of the interface is easily seen to be

$$\tfrac{1}{4}\alpha^2(F+mc^2)\,\hat{\eta}^2 = \tfrac{1}{4}m\alpha^2(c_0^2+c^2)\,\hat{\eta}^2. \tag{3.11}$$

An expression is established in the Appendix for the kinetic-energy loss W experienced by the fluid when the wave is created in the conservative system. Subtracting this from (3.11) we obtain the following expression for the total-energy level, which with c_r substituted for c gives the activation energy in the non-conservative system:

$$E = \tfrac{1}{4}\rho\alpha\{\zeta(c_0^2+c^2)-(U_0^2-c^2)\}\,\hat{\eta}^2. \tag{3.12}$$

When $c \equiv c_r$ is eliminated by means of (3.9), this gives

$$E = \frac{\rho\alpha}{2(\zeta+1)}\,u(u\mp U_0)\,\hat{\eta}^2, \tag{3.13}$$

where the alternative sign is to be chosen as in (3.9). [We note that the principle on which this derivation is based, namely that E is equivalent to the energy level in a corresponding conservative system, generalizes the interpretation of a result given in Landahl's paper. It appears that the second term in his equation (64) is a general expression for the activation energy, the contributory properties of the solid and fluid being represented implicitly by the mechanical impedance of the interface in the presence of the flow.]

The net rate of irreversible energy-conversion by the disturbance is $\overline{\sigma\partial\eta/\partial t}$, where $\sigma = (\rho S - km\alpha)(\partial\eta/\partial t)$ is the resultant normal stress in phase with the normal velocity of the interface. Thus we find that

$$\frac{dE}{dt} = \tfrac{1}{2}\rho\alpha^2c^2(S-\zeta k)\,\hat{\eta}^2. \tag{3.14}$$

This expression corresponds to (2.8), being the difference between the mean rate of energy-transfer due to the non-conservative part of the hydrodynamic pressure and the mean rate of dissipation.

† It may be of interest to note the case of 'internal waves' at the interface between superposed fluids, the lower of which has a density ρ_w only slightly in excess of the density ρ of the upper (see the remarks in brackets just below (3.5)). One then has $\zeta = \rho_w/\rho \doteqdot 1$, so that the value $U_0^2 = (\zeta-1)\,g/\alpha$ at which class A instability can arise is approximately *half* the value $U_0^2 = (\zeta-\zeta^{-1})\,g/\alpha$ for Kelvin–Helmholtz instability (cf. Lamb 1932, § 232, equation (12)).

In case (i) where $u > U_0$ the activation energy E is positive according to (3.13), both for the waves with $c = a$ and for those with $c = b$. Hence the waves grow in amplitude when $dE/dt > 0$, which (3.14) shows to be when $S > \zeta k$. In keeping with the general arguments set out in § 2, these facts clearly explain the destabilizing effect of irreversible energy-transfer from the fluid (i.e. of Miles's energy-transfer mechanism) and the stabilizing effect of dissipation. But note that dE/dt is not the rate of accumulation of energy by the solid, although it is approximately when $\zeta \gg 1$ (the 'Jeffreys–Miles approximation').

In case (ii) where $0 < u < U_0$ equation (3.13) shows that E is again positive for the class B waves with $c = b$, but is negative for the class A waves with $c = a$. Since growth of the latter waves requires that $dE/dt < 0$, the reversed roles of irreversible energy-transfer and dissipation are thus explained in accord with the general interpretation of class A behaviour given in § 2.

The condition for incipience of class C instability is that $u = 0$, in which case $E = 0$ independently of the value of $\hat{\eta}$. This result reflects the essential physical mechanism of this type of instability as was explained in § 2, namely that a conservative exchange of energy takes place between the two parts of the system, so leaving the activation energy virtually unchanged.

The antecedent of the material of this paper in the published work of Prof. Marten T. Landahl has already been made clear, and I wish to acknowledge also my indebtedness to him for inspiring discussions of the present subject.

Appendix. The kinetic-energy level of an infinite flow disturbed by a travelling wave

To find the energy level relative to the undisturbed state, the simplest course is to consider a perturbation from the plane boundary in the form

$$y = \eta(x,t) = f(t)\, e^{i\alpha(x-ct)}, \tag{A 1}$$

with α and c real, and allow the amplitude $f(t)$ to increase over a finite time from zero up to its final steady value $\hat{\eta}$. For the present purpose the undisturbed flow may be taken to have a uniform velocity U_0. The disturbed motion is therefore irrotational, and the velocity potential satisfying the kinematical condition at the boundary (A 1) is found to be

$$\phi = -\left\{\frac{\dot{f}}{\alpha} + i(U_0 - c)\, f\right\} e^{i\alpha(x-ct)-\alpha y}. \tag{A 2}$$

Hence the pressure on the boundary is, by Bernoulli's theorem,

$$\begin{aligned} p_s &= -\rho\left(\frac{\partial\phi}{\partial t} + U_0\frac{\partial\phi}{\partial x}\right)_{y=0} \\ &= \rho\left\{\frac{\ddot{f}}{\alpha} + 2i(U_0 - c)\,\dot{f} - \alpha(U_0 - c)^2 f\right\} e^{i\alpha(x-ct)}. \end{aligned} \tag{A 3}$$

Averaged over x, the rate at which the fluid does work on the boundary (i.e. the rate at which it loses energy) is

$$dW/dt = -\overline{p_s \partial\eta/\partial t} = -\tfrac{1}{2}\,\mathrm{Re}\,\{(p_s)^* \,(\partial\eta/\partial t)\}, \tag{A 4}$$

where the parentheses refer to the complex amplitudes of the enclosed quantities and the asterisk denotes a complex conjugate. Thus we obtain

$$dW/dt = -\tfrac{1}{2}\rho\{\alpha^{-1}\dot{f}\ddot{f} - \alpha(U_0^2 - c^2)f\dot{f}\}. \qquad \text{(A 5)}$$

It may be supposed that a steady wave is created by some process within the boundary, which starts with $f = \dot{f} = 0$ and concludes with $f \to \hat{\eta}, \dot{f} \to 0$. Then by integration of (A 5) we find that the *loss* of kinetic energy by the fluid, per unit surface area of the boundary, is

$$W = \tfrac{1}{4}\alpha(U_0^2 - c^2)\hat{\eta}^2, \qquad \text{(A 6)}$$

in agreement with Landahl (1962, equation (60)).

REFERENCES

Benjamin, T. Brooke 1959 Shearing flow over a wavy boundary. *J. Fluid Mech.* **6**, 161.

Benjamin, T. Brooke 1960 Effects of a flexible boundary on hydrodynamic stability. *J. Fluid Mech.* **9**, 513.

Benjamin, T. Brooke 1961 Dynamics of a system of articulated pipes conveying fluid. I. Theory. *Proc. Roy. Soc.* A, **261**, 457.

Betchov, R. 1960 Simplified analysis of boundary-layer oscillations. *J. Ship Res.* **4**, no. 2, p. 37.

Boggs, F. W. & Tokita, N. 1960 A theory of the stability of laminar flow along compliant plates. *Proc. Third Symp. on Naval Hydrodynamics, Scheveningen, Holland.*

Hains, F. D. & Price, J. F. 1962 Effect of a flexible wall on the stability of Poiseuille flow. *Phys. Fluids*, **5**, 365.

Hawthorne, W. R. 1961 The early development of the Dracone flexible barge. *Proc. Instn Mech. Engrs*, **175**, 52.

Hedgepeth, J. M., Budiansky, B. & Leonard, E. W. 1954 Analysis of flutter in compressible flow of a panel with many supports. *J. Aero Sci.* **21**, 475.

Johns, D. J. & Parks, P. C. 1960 Effect of structural damping on panel flutter. *Aircraft Eng.* **32**, 304.

Krämer, M. O. 1960 Boundary layer stabilization by distributed damping. *J. Amer. Soc. Nav. Engrs*, **72**, 25; *J. Aero/Space Sci.* **27**, 68.

Lamb, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.

Landahl, M. T. 1962 On the stability of a laminar incompressible boundary layer over a flexible surface. *J. Fluid Mech.* **13**, 609.

Lighthill, M. J. 1960 Note on the swimming of slender fish. *J. Fluid Mech.* **9**, 305.

Lighthill, M. J. 1962 Physical interpretation of the mathematical theory of wave generation by wind. *J. Fluid Mech.* **14**, 385.

Miles, J. W. 1957 On the generation of surface waves by shear flows. *J. Fluid Mech.* **3**, 185.

Miles, J. W. 1959 On the generation of surface waves by shear flows. Part 2. *J. Fluid Mech.* **6**, 568.

Miles, J. W. 1962*a* On the generation of surface waves by shear flows. Part 4. *J. Fluid Mech.* **13**, 433.

Miles, J. W. 1962*b* Generation of surface waves by shear flows. *Proc. Symposia in Appl. Math.* **13**, 79.

Nelson, H. C. & Cunningham, H. J. 1955 Theoretical investigations of flutter of two-dimensional flat panels with one surface exposed to supersonic potential flow. *N.A.C.A. Tech. Note*, no. 3465.

Nonweiler, T. 1961 Qualitative solutions of the stability equations for a boundary layer in contact with various forms of flexible surface. *A.R.C. Rep.* no. 22,670.

Schlichting, H. 1955 *Boundary Layer Theory.* London: Pergamon.